EFFICIENT ESTIMATION
OF RADAR ASTRONOMY TARGET PARAMETERS

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One of the objectives of radar astronomy is to use the radar signal scattered from the target to estimate some target parameter of interest (e.g., the rate of rotation of a planet). The usual situation is one in which a long observation time still yields estimates whose accuracy is no more than adequate. Two (usually conflicting) criteria are thus of paramount importance for any data processing method used to obtain estimates of the target parameters: computational simplicity and asymptotic efficiency. In this paper we present a recursive estimation algorithm which is quite simple from a computational standpoint, and yet is asymptotically efficient for a broad class of problems.

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We are concerned with the radar (or radio) astronomy situation in which one sequentially observes a signal W(t) on successive time intervals $I_1, I_2, I_3, \cdots$, each of duration T. A reasonable model for this situation is $W(t) = N(t) + S(t)$ in which N(t) is White Gaussian zero-mean noise of power spectral density N and S(t) is a zero-mean Gaussian process scattered (or originating) from the target. We assume that observations of W(t) made on the successive intervals I_1, $I_2, \cdots$, are all identically distributed and statistically independent.

Our interest here is focused on the case when the form of the target is known and its scattering (or radiating) properties are determined by M parameters whose values are unknown to us. We describe the possible values assumed by these parameters by the M-dimensional vector α. Our objective is to estimate $\underline{\alpha}$, the actual value of the parameters. From the assumed form of the target and a knowledge of the transmitted signal, we can calculate the functional dependence of the correlation function of S(t) on α; we denote this function by $\phi_s(t, s, \underline{\alpha})t, s \in [0, T]$ i.e.,

$$\phi_s(t, s, \underline{\alpha}) = E\{S_t S_s\} \ .$$

We wish to use this functional dependence together with the sequence of observations to estimate $\underline{\alpha}$. While there are many ways of achieving this, two considerations are paramount: the amount of data to be handled is extraordinarily large and yet the estimation error that can be obtained is no more than satisfactory. This leads one to attempt to find estimation methods which are computationally convenient and yet asymptotically efficient.

Our work consists of two parts. In the first, we give explicit expressions for the partial derivatives of the log-likelihood function and an expression for the matrix appearing in the Cramér-Rao bound on the estimation error. In the second, we describe a (computationally) convenient recursive estimation procedure and indicate conditions under which this method is asymptotically efficient.

Let $h(t, s, \underline{\alpha})$ denote the kernal of the optimum detection operation first derived by Price [1] for the signal with correlation function $\phi_s(t, s, \underline{\alpha})$; i.e., $h(t, r, \underline{\alpha})$ is the solution of

$$N h(r, t, \underline{\alpha}) + \int_0^T \phi_s(r, s, \underline{\alpha}) h(s, t, \alpha) \, ds$$

$$= \phi_s(r, t, \underline{\alpha}) r, t \in [0, T] \ .$$

Let $\phi_s^i(t, s, \underline{\alpha})$ and $h^i(t, s, \underline{\alpha})$ denote the partial derivatives with respect to α_i of $\phi_s(t, s, \underline{\alpha})$ and $h(t, s, \underline{\alpha})$ respectively.

If we denote by $\ell(\underline{\alpha})$ the log-likelihood function of α based on an observation of W(t) of duration T, we have shown that

$$Y_i(\underline{\alpha}) = \frac{\partial \ell(\underline{\alpha})}{\partial \alpha_i} = \frac{1}{2N} \int_0^T \int_0^T h^i(t, s, \underline{\alpha}) \cdot$$

$$\left[W(t)W(s) - \phi_w(t, s, \underline{\alpha})\right] dt ds$$

in which $\phi_w(t, s, \underline{\alpha}) = N\delta(t-s) + \phi_s(t, s, \underline{\alpha})$. It is known that if $\hat{\underline{\alpha}}$ denotes any unbiased estimate of $\underline{\alpha}$ based on n independent observations of W(t), each of duration T, then

$$\underline{c}' \, \underline{\Sigma} \, \underline{c} \geq \underline{c}' B^{-1}(\underline{\alpha}) \underline{c} \ (1/n)$$

in which ' denotes transpose, $\underline{c}$ is an arbitrary vector, Σ is the matrix whose i-jth element is

$$E\left\{(\hat{\alpha}_i - \alpha_i)(\hat{\alpha}_j - \alpha_j)\right\}$$

and $B(\underline{\alpha})$ is the matrix whose i-jth element is

$$b_{ij}(\underline{\alpha}) = E\left\{Y_i(\underline{\alpha}) Y_j(\underline{\alpha})\right\}$$

We have shown that the $b_{ij}(\theta)$ are equal to

$$\frac{1}{2N}\int_0^T\int_0^T h^i(t, s, \theta)\phi_s^j(t, s, \theta)\,dt\,ds$$

for $i, j = 1, 2, \cdots, M$.

Having given an expression for the minimum possible estimation error, we now describe a recursive estimation method for using the sequential observations and indicate conditions under which this method asymptotically achieves this lower bound. Let $Y_n(\theta)$ denote the M-dimensional vector valued observation whose i-th component is $Y_i(\theta)$ evaluated on the n-th observation interval I_n. We will denote

$$m(\theta) = E\left\{Y_n(\theta)\right\}.$$

It is assumed that we know a priori that θ lies in the interior of some bounded rectangle A, and we restrict our estimation to this set. We denote by $H(\theta)$ the matrix whose i-jth element is

$$\frac{1}{2N}\int_0^T\int_0^T h^i(t, s, \theta)\phi_s^j(t, s, \theta)\,dt\,ds$$

(note that $h_{ij}(\theta) = b_{ij}(\theta)$). We assume that for $\theta \in A$, $H(\theta)$ is invertible, and denote by $g_{ij}(\theta)$ the elements of the inverse, $G(\theta)$.

Our recursive procedure generates a sequence of estimates $\theta_1, \theta_2, \cdots$, as follows. We select θ_1 arbitrarily in A, and at the end of the n-th observation interval I_n form the vector

$$\eta_{n+1} = \theta_n - (1/n)G(\theta_n)Y(\theta_n) \quad n=1,2,\cdots$$

If the j-th component of η_{n+1} lies in A, we take $\alpha_{n+1, j} = \eta_{n+1, j}$, otherwise, we take $\alpha_{n+1, j}$ to lie on the appropriate boundary of A.

We wish to make a statement concerning the convergence of α_n to θ. In addition to the conditions already stated, two more are required. The first of these is simply a regularity condition and will not be stated here. The second condition is not so trivial, and is essentially the condition that delineates when our method is applicable. We assume that there exist K_0 and K_0' with

$$0 \le K_0 \le K_0' \le \infty \quad \text{and}$$

$$K_0\|\alpha - \theta\|^2 \le (\alpha - \theta)'G(\alpha)m(\alpha)$$

$$\le K_0'\|\alpha - \theta\|^2.$$

Let us comment briefly on this condition. $G(\theta)$ is the inverse of the matrix of partial derivatives of the components of $m(\theta)$ at $\alpha = \theta$. Thus this condition holds at $\alpha = \theta$ with $K_0 = K_0' = 1$ and will always hold in the weakened form above in some region about $\alpha = \theta$. In an example, we have considered the case in which $S(t)$ is a Markoff process of unknown variance and time-constant. In this situation the above condition holds for all possible values of these two unknowns that are strictly positive.

Under the conditions stated, we have shown that the convergence of α_n to θ is such that

$$E\left\{(\alpha_{n,i} - \theta_i)^2\right\} \le g_{ii}(\theta)(1/n)+0(1/n^{1+\gamma})$$

$$\gamma > 0.$$

Thus we asymptotically achieve the bound given by the Cramér-Rao inequality.

Reference

[1] Price, R., Optimum Detection of Random Signals in Noise with Application to Scatter-Multipath Communication, IRE Trans., IT-2, No. 4, pp. 125; 1956.